Controlling unstable states in reaction-diffusion systems modeled by time series

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We introduce an algorithm for controlling unstable states of a spatiotemporal system modeled by a time series. Control is achieved by adjusting an external parameter at the boundary. Our time series is taken as the concentration from an experiment modeled by a reaction-diffusion system. It is shown that unstable states can be maintained by performing fluctuations of the concentration at the boundaries, while monitoring the dynamics from an interior spatial point.

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I. INTRODUCTION

Recent developments in the theory of nonlinear dynamical systems have provided experimentalists with new tools for exploring a wide range of aspects in the dynamics of real systems based on analyzing a single time series. Methods such as embedding techniques allow one to reconstruct the geometric model of the attractor and recover all its essential properties from time series measurements alone [1,2]. These methods have led to a control algorithm [known as the Ott-Grebogi-Yorke (OGY) algorithm)] for stabilizing unstable orbits inside a chaotic attractor [3] by applying small, carefully computed perturbations of an accessible system parameter.

Recently, the authors have designed an algorithm that stabilizes unstable orbits and also tracks them as a function of a system parameter, thus extending the region over which control can be achieved [4]. This algorithm also applies to the time series itself and makes use of embedding techniques. As the parameter is varied, control is maintained by a predictor-corrector technique. The correction step incorporates the OGY technique or any analogous form of linear control. The tracking algorithm was implemented for maps as well as flows, and has been successfully applied to experiments [5,6].

Tracking an unstable state of spatiotemporal processes usually modeled by partial differential equations is also possible and will constitute the subject of a future paper. Tracking and control along unstable branches as a function of a parameter can lead to interesting new stable patterns that do not form spontaneously in an experiment [8], possibly leading to new experimentally realizable regimes. As a first step in this direction, we present a method of stabilizing an unstable state, which achieves control spatially as well as temporally. The method is applied to a reaction-diffusion system that models pattern formation in Couette flow reactors. This system exhibits both small amplitude chaos and chaotic bursting [7]. Our goal is to stabilize an unstable periodic orbit when the dynamics exhibits periodic bursting or chaotic behavior.

In our method, the numerical solution of the system is generated by a partial-differential-equation (PDE) solver, and control is applied by adjusting the boundary data referencing the dynamics at a fixed spatial point. We simulate the solution from an experiment where only the time series at a spatial point is accessible.

II. MODEL

We consider the following one-dimensional reaction-diffusion system:

$$\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}+\frac{1}{\epsilon}[v-f(u)]\ ,$$
$$\frac{\partial v}{\partial t}=D\frac{\partial^2 v}{\partial x^2}-u+\alpha,\quad x\in[0,1]\ , \tag{1a}$$

subject to Dirichlet boundary conditions,

$$u(x=0,0)=u_0,\quad v(x=0,0)=v_0\ ,$$
$$u(x=1,0)=u_0,\quad v(x=1,0)=v_0\ . \tag{1b}$$

The reaction term is a two-variable Van der Pol–like equation, which accounts for the excitable bursting character of the dynamics. We remark that in the absence of diffusion, chaotic solutions are not possible.

This is a formal model that does not completely meet the experimental conditions and the requirements of chemical kinetics laws in the Couette flow reactor. However, it reproduces most of the phenomena associated with the observed front patterns in a chlorite-iodide reaction [7]. The interaction of reaction and diffusion terms gives rise to a variety of sustained patterns such as stationary periodic structures, nonlinear waves, or chaotic spatiotemporal structures of large amplitude. In [9], numerical evidence for chaotic intermittent bursting was reported and analyzed taking the transport rate D as a bifurcation parameter. In our study, we noticed extreme sensitivity of the solution with respect to the diffusion coefficient D as well as with respect to α, which did not allow the use of these parameters for control. (Changes in the seventh significant digit were sufficient to change the attractors.) Instead we looked at the solution as one of the Dirichlet boundary conditions is varied and observed transition to chaos via an intermittency route, the details of which will be presented elsewhere.

III. ALGORITHM

The algorithm we present is meant for stabilizing unstable orbits of a spatiotemporal process modeled by a time series; a desired unstable orbit is maintained by adjusting an external system parameter. For Eqs. (1a) and (1b), the time series is measured at one spatial point in the interior region. Control is achieved by appropriately choosing fluctuations in the boundary conditions, introduced as one integrates in time. In fact, it is sufficient to use only one of the Dirichlet boundary conditions as an accessible parameter, which we will refer to from now on as our control parameter.

In order to maintain the system on the unstable state, we measure a time series at $x=x_0$, namely $v(x_0,t)$. The fixed space value is taken anywhere in the middle region of the interval (approximately the middle third of the interval) where the most severe bursting occurs. The system is reaction dominated in this region.

One way to form a discrete dynamics of the time series is sampling the variable v at successive minima. (Due to the strong coupling between the variables u and v, control applied to the v variable leads to controlling the u variable as well.) If we denote successive minima in the time series for $v(x,t)$ by v_n, we obtain a map denoted by f:

$$v_{n+1}=f(v_n,p)\ , \tag{2}$$

where p stands for the control parameter, which in our case is one of the boundary conditions. An unstable orbit of this map is then controlled by using any of the linear control methods.

Control is extended in time as follows. At each iterate of the map (2), the computed value of the solution v_n is used to determine the fluctuation δp_n in the parameter according to a linear control method. The change in the parameter will be proportional to the deviation from the unstable state to be controlled. The new value of the parameter $p_n+\delta p_n$ is fed back into the PDE solver. The solution v_{n+1} at $p_n+\delta p_n$ is obtained, and a new evaluation of the fluctuation in the boundary condition then follows based on v_{n+1}, and so on. Control of the unstable orbit is thus extended in time.

To fix ideas, we assume from here on that the state we are interested in is a period-1 fixed point v_0 of the map f; i.e., $v_0=f(v_0,p)$. Such a fixed point corresponds to a period-1 time series at the spatial point x_0.

As our linear control algorithm, we used the OGY control method, which amounts to ensuring at each time iteration that the next iterate of the map will fall on the stable manifold of the unstable state we are controlling. In the case of a two-dimensional map, if we denote by λ_s,λ_u the eigenvalues of the unstable state v_0 and by f_u the contravariant vector corresponding to the unstable direction, then the above mentioned condition applied to the linear approximation of the map yields that the control parameter p must be modified by

$$\delta p_n \equiv \frac{\lambda_u[\mathbf{v}_n-\mathbf{v}_0(p)]\cdot\mathbf{f}_u}{(\lambda_u-1)\mathbf{g}\cdot\mathbf{f}_u}\ , \tag{3}$$

at each iteration of the map [3]. Notice that Eq. (3) depends on the spatial point at which the time series is measured. The vector g is the derivative of the unstable state v_0 with respect to p. In the case of weak diffusion, the map is in fact nearly one dimensional at a spatial point. This amounts to having $\lambda_s=0$, in which case the formula becomes

$$\delta p_n \equiv \frac{\lambda_u[\mathbf{v}_n-\mathbf{v}_0(p)]}{(\lambda_u-1)\mathbf{g}}\ . \tag{4}$$

Equation (4) is a traditional control method known as occasional proportional feedback [10,11]. The eigenvalues and eigenvectors involved in the formulas (2), (3), or (4) associated with the saddle $v_0(p)$ can all be calculated based on the reconstructed attractor.

Summarizing, our method achieves control of a spa-

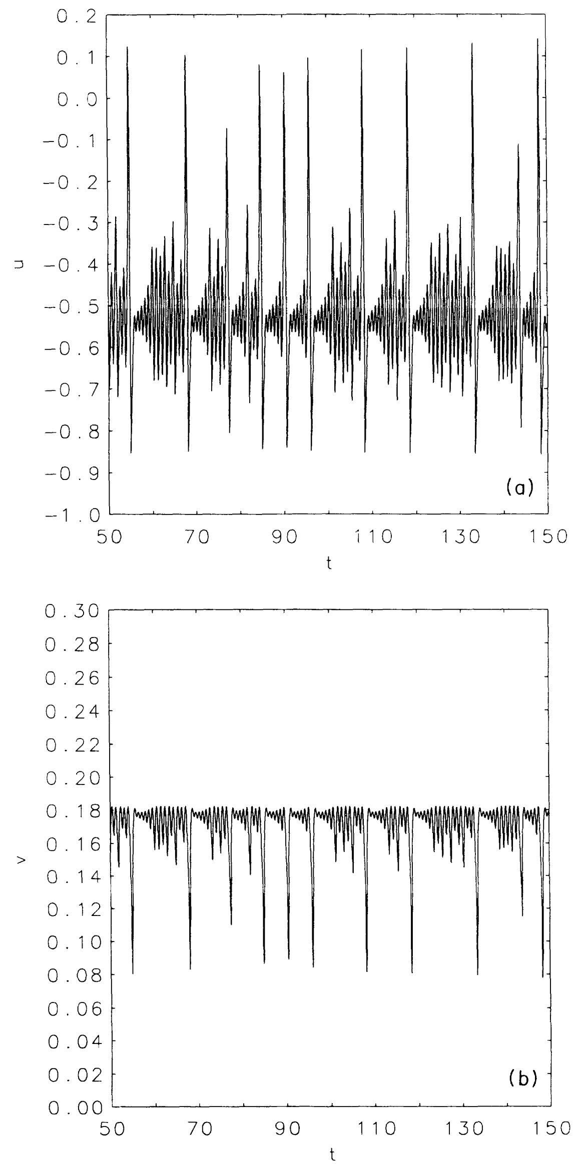

FIG. 1. (a) and (b) Time series for u and v, respectively, recorded at $x=\frac{1}{2}$ and $p=-2.0$ in the absence of control.

tiotemporal process as follows. After reconstructing a discrete map from the data sampled at a fixed spatial point, control is implemented by fluctuating the boundary conditions proportional to the deviation from the state we want to maintain. The method designed in this way has the advantage in that it can be applied directly to experimental data for which an analytical model is unavailable, and requires no mode expansion as in [12].

IV. NUMERICAL RESULTS

In our numerical tests we took in the model (1a) $f(u)=u^2+u^3$, as in [7]. Before illustrating our scheme, we examined the bifurcation of solutions by taking the control parameter to be $p=u(0,t)$. Let us take $D=0.032\,249$ and $\alpha=0.01$, values at which a stable period one orbit exists for p between $p=-0.5$ and $p=-0.6$. As p is decreased past -0.6 the period-1 orbit becomes unstable and bifurcates into a period-2 orbit. At about $p=-1.15$ the period two destabilizes, giving rise to an intermittent bursting regime, which becomes chaotic as we further decrease p past $p=-1.6$.

Figures 1(a) and 1(b) show the time series for the solution at $x=\frac{1}{2}$ and $p=-2.0$, without control. The time series is chaotic and exhibits three distinct types of oscillations. First there is a large amplitude burst, which occurs on a fast reaction time scale. Following the burst, there occurs an exponentially growing small amplitude oscillation. This is followed by small amplitude chaotic oscillations whose length is random in time.

For a time series of 14 000 points at $x=\frac{1}{2}$, we compute an information dimension of 2.1, having one positive Lyapunov exponent [14]. Spectral analysis reveals that most of the energy is contained in four spatial modes. Note that time series sampled at other spatial points may have no positive Lyapunov exponent. We have not found any other chaotic solutions having more than one positive Lyapunov exponent.

From the solution, we form a map for which the iterates are the successive minima of the variable v. Figure 2 shows the successive minima of v when $D=0.032\,249$, $\alpha=0.01$, and $p=-2.0$, values of the parameters at which the solution exhibits chaos. Simultaneously, we display a stable period-6 solution in the intermittent regime at $D=0.032\,249$, $\alpha=0.01$, and $p=-1.4$. From the picture we see that this orbit obeys nearly the same nonlinear law as the chaotic attractor at $p=-2.0$. Similar results of periodic bursting hold in the Belovsov-Zhabotinsky (BZ) continuously stirred tank reactor [13]. The chaotic map in Fig. 2 is nearly one dimensional, justifying our use of the occasional proportional feedback in

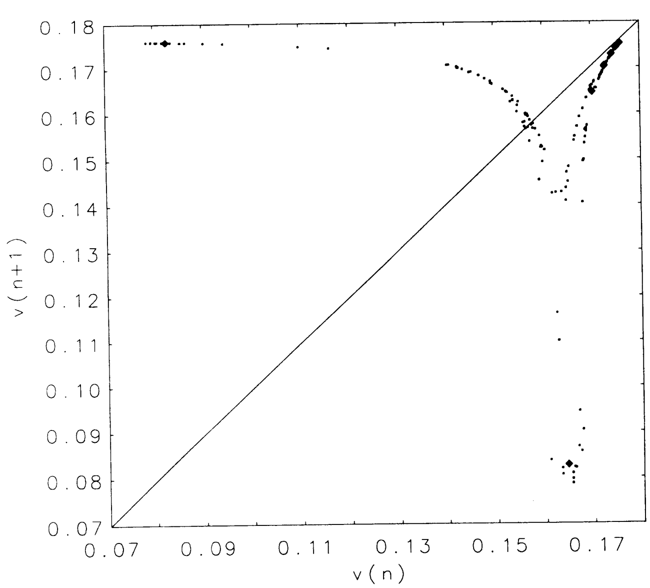

FIG. 2. v_{n+1} vs v_n at $D=0.032\,249$, $\alpha=0.01$, and $p=-2$. Large dots indicate the period-6 orbit at $p=-1.4$. Notice that both parameter values of p obey nearly the same dynamics law.

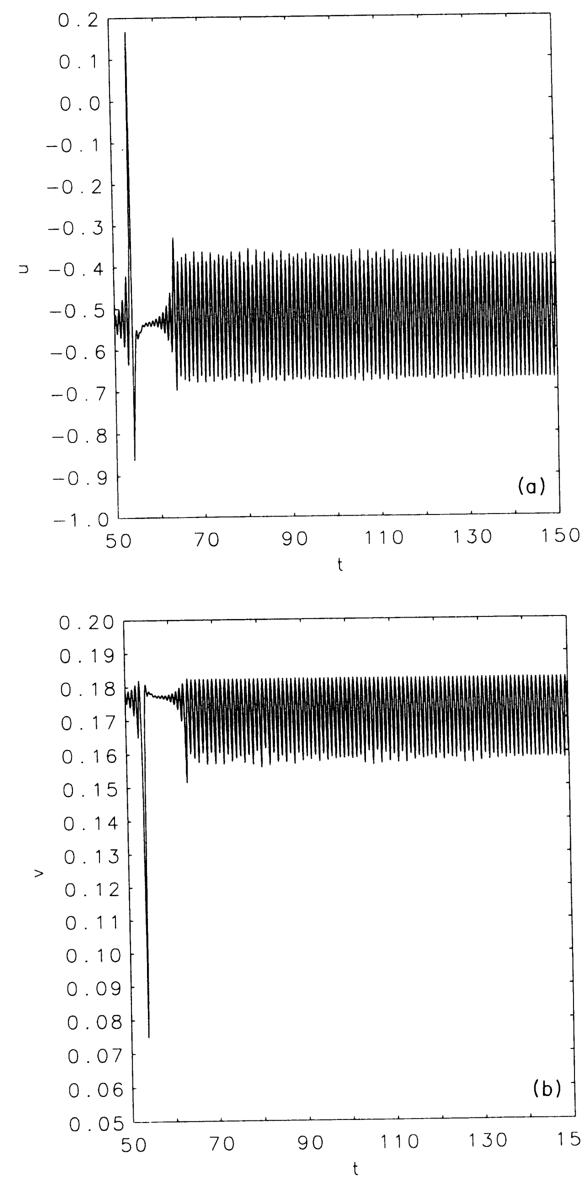

FIG. 3. (a) and (b) Time series for u and v, respectively, recorded at $x=\frac{1}{2}$ and $p=-1.4$, when control is applied to the corresponding time series at $x=\frac{1}{3}$.

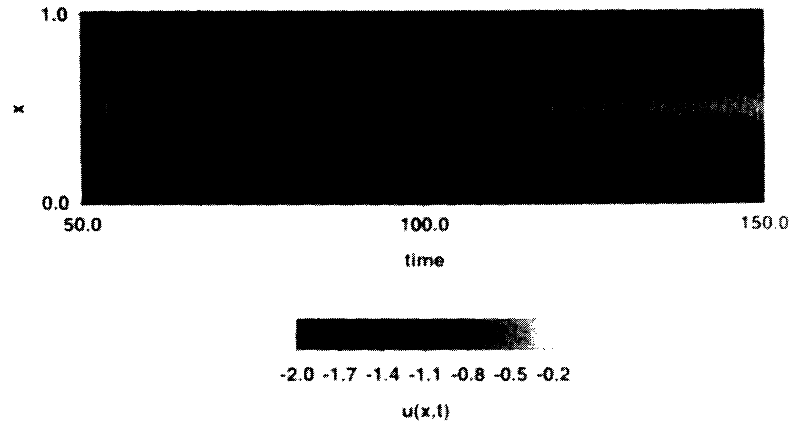

FIG. 4. Spatiotemporal pattern when control is applied to the time series at $x=\frac{1}{3}$.

our algorithm. Notice that the three oscillation types are clearly evident in the one-dimensional attractor reconstruction, the large bursts are near the peak of the map, the exponentially growing solutions form the beginning of the left branch, and the small amplitude chaos is formed around the period-1 fixed point. Also notice that at $p=-1.4$, the period-1 fixed point is inaccessible, since the stable period 6 is attracting.

Since control of period-1 orbits in chaotic attractors has been done elsewhere [11], we now describe control of inaccessible period-1 points. We consider a parameter value $p=-1.4$, where a stable periodic intermittent bursting solution exists. This orbit consists of a large burst followed by five small growing oscillations and is shown on the graph mapped in Fig. 2. We remark that at $p=-1.4$, the period-1 fixed point and its local neighborhood are inaccessible. That is, the dynamics does not enter a neighborhood of the period-1 fixed point, since there is no chaos. For control to occur, we fluctuate the parameter so that the dynamics enters the neighborhood of the period-1 fixed point. We do this by looking at the first preimage of a controllable neighborhood of the fixed point in the bursting regime, on the right branch of the map. The parameter is adjusted at the bursts so the dynamics enters a neighborhood of the period-1 fixed point, at which point control is implemented.

We generate a time series by sampling at $x=\frac{1}{3}$. In order to choose the reference value $v_0(p)$ at $p=-1.4$ in formula (4), we notice that the orbit we want to control has a fixed point that lies on the $y=x$ line in Fig. 2. Because the dynamics at $p=-1.4$ is approximated by the map at $p=-2$ (as shown in Fig. 2), we approximate $v_0(-1.4)$ by $v_0(-2)$.

Figures 3(a) and 3(b) show the stabilized solution at $x=\frac{1}{2}$, when the control is based on the solution sampled at $x=\frac{1}{3}$. The amplitude of this solution agrees with the amplitude of the stable solution at $p=-0.5$, where the period-1 orbit is stable.

During control of the periodic solution, the amplitude of the control at the boundaries is approximately 30% of the signal at $x=\frac{1}{2}$. The control perturbations are larger than in previous applications [4,5], since they must overcome weak diffusion to be effective in the interior.

In Fig. 4 the whole stabilized spatiotemporal pattern is shown. The period-1 solution in this example is indeed unstable. To see this, in Fig. 5 we show the same time series as in Fig. 3 where control was removed after 200 iterates. This results in the reappearance of the intermittent pattern after a short delay. In this example, the time series was sampled at $x=\frac{1}{2}$ and we started with an initial condition that is constant and equals the boundary conditions at $x=0$. The control parameters were not optimized to minimize fluctuations about the period-1 reference state.

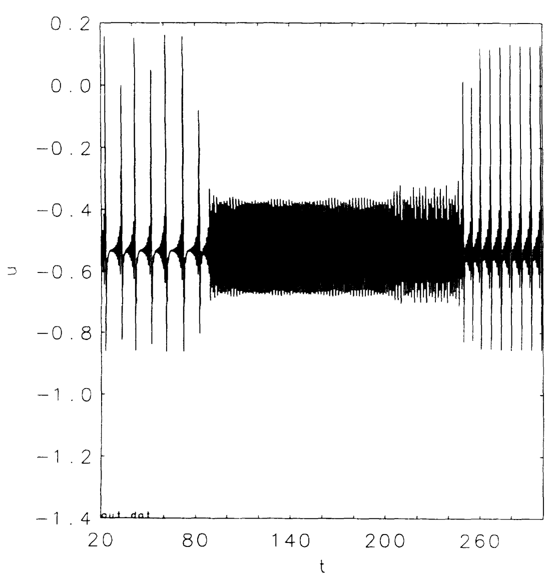

FIG. 5. Time series for u, respectively recorded at $x=\frac{1}{2}$ and $p=-1.4$, when control is applied at $x=\frac{1}{2}$ for the first 200 iterates. Control is removed afterwards.

V. CONCLUSIONS

We have introduced and tested an algorithm that applies to stabilizing unstable states of spatiotemporal processes occurring in reaction-diffusion processes. The novelty of this procedure consists in the fact that it applies to the time series directly, combining nonlinear analysis embedding techniques with classical linear control. Compared to similar techniques, it has the advantage that it achieves control spatially as well as temporally. Since it applies to the time series directly, the method is suitable for experimentalists. Furthermore, by using the targetting of intervals in periodic regimes, previously inaccessible control points are now achievable by making use of the global nonlinear dynamics.

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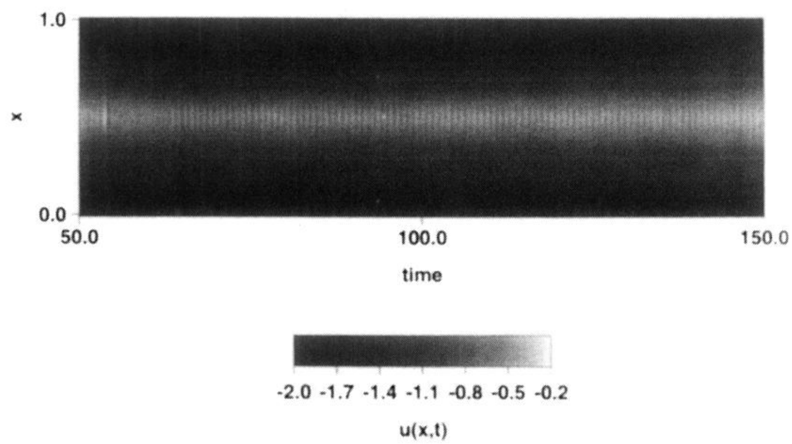

FIG. 4. Spatiotemporal pattern when control is applied to the time series at $x = \frac{1}{3}$.

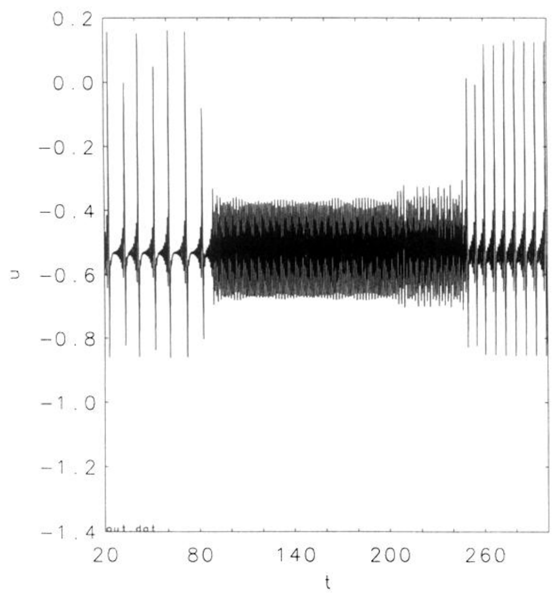

FIG. 5. Time series for u, respectively recorded at $x=\frac{1}{2}$ and $p=-1.4$, when control is applied at $x=\frac{1}{2}$ for the first 200 iterates. Control is removed afterwards.